Wave Guide.

1. Group velocity $\rightarrow$ if a region consists of 2 or more wave trains, the physical vel. of propagation of waves is gr. vel.

Phase vel. $\rightarrow$ vel. of individual wave

$$\boxed{v_p = \frac{\omega}{K}}$$

(~~vel.~~ vel. of the point of const. phase)

(not vel. of propagation of any physical quantity)

Signals are transmitted with gr. vel. V_G

$$\boxed{V_G = \frac{\partial \omega}{\partial K}}$$

Derivation

$$E_1 = E_0 \cos \omega_1 \left(t - \frac{z}{v_1}\right)$$
$$E_2 = E_0 \cos \omega_2 \left(t - \frac{z}{v_2}\right)$$

$v_1, v_2 \rightarrow$ phase vel.

$$E = E_1 + E_2 = E_0 \left(\cos \omega_1 \left(t - \frac{z}{v_1}\right) + \cos \omega_2 \left(t - \frac{z}{v_2}\right)\right)$$
$$= 2E_0 \cos \left\{\frac{\omega_1}{2}\left(t - \frac{z}{v_1}\right) + \frac{\omega_2}{2}\left(t - \frac{z}{v_2}\right)\right\}$$
$$\cos \left\{\frac{\omega_1}{2}\left(t - \frac{z}{v_1}\right) - \frac{\omega_2}{2}\left(t - \frac{z}{v_2}\right)\right\}$$

Let $\omega_1 = \omega$

$\omega_2 = \omega_1 + d\omega = \omega + d\omega$

$\omega_1 - \omega_2 = -d\omega$, $\quad \omega_1 + \omega_2 = 2\omega + d\omega \approx 2\omega$

~~$E = 2E_0 \cos\omega(\omega t -$~~

$$v = \frac{v_1 + v_2}{2} \qquad \frac{\omega_1}{v_1} + \frac{\omega_2}{v_2}$$

$$= \frac{\omega_1}{v_1} + \frac{\omega_1 + d\omega}{v_2}$$

$$= \frac{\omega_1 v_2 + \omega_1 v_1 + d\omega\, v_1}{v_1 v_2}$$

$$= \frac{\omega_1 (v_1 + v_2) + d\omega\, v_1}{v_1 v_2}$$

$$= \frac{\omega_1 2v + d\omega (v_1)}{v_1 v_2}$$

$$= \frac{2v\omega_1 + v_1 d\omega}{v_1 (2v - v_1)}$$

$$= \frac{2\cancel{v}\omega_1 + \cancel{v} d\omega}{\cancel{v}^2}$$

$$= \frac{2\omega_1}{v}$$

$$-\frac{z}{2}\left(\frac{\omega_1}{v_1}+\frac{\omega_2}{v_2}\right) = -\frac{2\omega z}{2v} = -\frac{\omega z}{v}$$

$$E = 2E_0 \cos\omega\left(t-\frac{z}{v}\right)\cos\left(-\frac{d\omega}{2}t + \frac{1}{2}\left(\frac{\omega_2}{v_2}-\frac{\omega_1}{v_1}\right)z\right)$$

Now $\frac{\omega_2}{v_2} - \frac{\omega_1}{v_1} = \frac{\omega+d\omega}{v_2} - \frac{\omega}{v_1}$

$v_1 = v$
$v_2 = v + dv$

$$\frac{\omega+d\omega}{v+dv} - \frac{\omega}{v} = \frac{v(\omega+d\omega)-\omega(v+dv)}{v(v+dv)}$$

$$= \frac{v\,d\omega - \omega\,dv}{v^2}$$

$$= d\left(\frac{\omega}{v}\right)$$

$$E = 2E_0 \cos\omega\left(t-\frac{z}{v}\right)\cos\left\{-\frac{d\omega}{2}t + \frac{1}{2}d\left(\frac{\omega}{v}\right)z\right\}$$

$$= 2E_0 \cos\frac{d\omega}{2}\left[t - \frac{d(\omega/v)}{d\omega}z\right]\cos\omega\left(t-\frac{z}{v}\right)$$

So, this equation represents a wave travelling with phase vel. v and amplitude

$$2E_0 \cos\frac{d\omega}{2}\left(t-\frac{z}{v_g}\right)$$

where $v_g = \frac{d\omega}{d\left(\frac{\omega}{v}\right)} = \frac{d\omega}{dk}$

Since amplitude velocity is signal velocity $\rightarrow$ and it is group velocity

Propagation of e.m. wave betn parallel conducting planes

We know in good conductor e.m. field penetrates upto skin depth. ~~For a~~ perfect conductor behaves as a perfect reflector. No field penetrate within it. E, H = 0 within perfect conductor.

Now from b.c at interface continuous

$E_{1t} = E_{2t}$, $E_{1t} = E_{2t} = 0$

$B_{1n} = B_{2n}$.

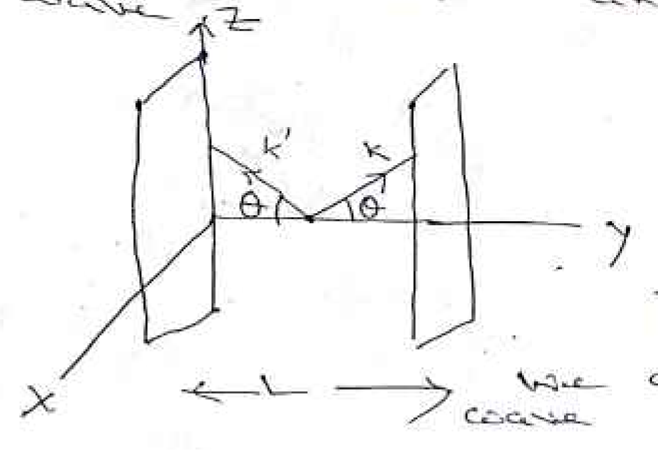
The other two condn B_{1t}, B_{2t}

$D_{1n} - D_{2n} = \sigma$ → surface charge density

$H_{1t} - H_{2t} = J_s$ → surface current

we need, σ, J → information.

Since no penetration, we consider only incident and reflected wave



Propagation of e.m. waves in region betn two perfectly conducting parallel plane.

we can consider wave in (y - z) plane

initial wave.

$$e^{i\vec{k}\cdot\vec{r}}$$
$$= e^{i\vec{k}\cdot(y\hat{j}+z\hat{k})}$$
$$= e^{ik\hat{n}\cdot(y\hat{j}+z\hat{k})}$$
$$= e^{ik(y\cos\theta+z\sin\theta)}$$

reflected wave

$$e^{i\vec{k}'\cdot\vec{r}} = e^{ik'\hat{n}'\cdot(y\hat{j}+z\hat{k})}$$
$$= e^{ik(-y\cos\theta+z\sin\theta)}$$

$e^{ik(y\cos\theta+z\sin\theta)-i\omega t}$ (incident)

$e^{ik(-y\cos\theta+z\sin\theta)-i\omega t}$ (reflected)

$|k| = |k'| = \frac{\omega}{c}$ (both are in free space)

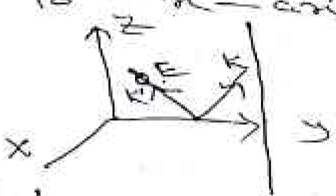
Now K, H, E are mutually perpendicular.

two possibilities

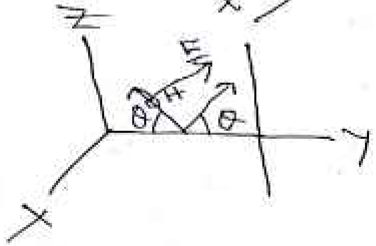
i) $\vec{E}$ ∥ to x-axis

$E_z = 0$

(TE)



ii)



$\vec{H}$ ∥ to x axis

(TM)

TE case

$$E_1 = \hat{i} E_{01} e^{ik(y\cos\theta + z\sin\theta) - i\omega t}$$ (incident wave)

$$E_1' = \hat{i} E_{01}' e^{ik(-y\cos\theta + z\sin\theta)}$$ (reflected wave)

$\hat{i} \rightarrow$ unit vector along x-axis.

By principle of superposition

$$E = E_1 + E_1'$$

$$= \hat{i} E_{01} e^{i(\;) - i\omega t} + \hat{i} E_{01}' e^{ik(\;) - i\omega t}.$$

b.c $= E_1 + E_1' = 0.$

$$0 = E_{01} e^{i(\;)} + E_{01}' e^{i(\;)}$$ at surface $[y = 0]$

$$E_{01} = -E_{01}'$$ $\rightarrow$ satisfied for all z, t

$$E = \hat{i} E_{01} \left(e^{iky\cos\theta} - e^{-iky\cos\theta} \right) e^{ikz\sin\theta} e^{-i\omega t}$$

$k = \frac{\omega}{c}$

$$= \hat{i} E_{01}\, 2i \sin(ky\cos\theta)\, e^{i(kz\sin\theta - \omega t)}$$

$$= \hat{i}\, 2i E_{01} \sin\left(\frac{\omega \cos\theta}{c} y \right) e^{i\left(\frac{\omega z \sin\theta}{c} - \omega t\right)}$$

This equation represent a disturbance propagating along z with velocity

$$v_p = \frac{c}{\sin\theta} > 1$$ $\boxed{ikz = i\frac{\omega z}{v_p}}$

since $\sin\theta < 1$.

but varying sinusoidally along y axis (corresponds to stationary wave along y-axis).

effective wavelength along y-axis λ_c.

$$k_c = \frac{2\pi}{\lambda_c} = \frac{\omega\cos\theta}{c}, \quad \lambda_c = \frac{2\pi c}{\cos\theta\, \omega}$$

Let $\lambda_0 = \frac{2\pi}{k} = \frac{2\pi}{\omega/c} = \frac{2\pi c}{\omega}$

(free space wavelength
(in absence of reflection))

$\therefore \boxed{\lambda_c = \frac{\lambda_0}{\cos\theta}}$

(microwave range)

Standing wave phenomena observed by Lippmann not in a free space but in a sensitive emulsion of photographic plate

So far we have consider b.c at $y = 0$,

Now consider $y = L$

$$E = i E_{01}\, 2i \sin\left(\frac{\omega \cos\theta}{c}\, y\right) e^{i\left(\frac{\omega \sin\theta}{c} z - \omega t\right)}$$

$$\sin \frac{\omega \cos\theta}{c} L = 0.$$

(b.c at $y = L$, $E_i = E_{r_0}$)

$$\frac{\omega \cos\theta}{c} L = n\pi$$

$\left(\lambda_c = \frac{2\pi c}{\omega \cos\theta}\right)$

$$L = n \frac{\pi c}{\omega \cos\theta} = n \frac{\lambda_c}{2}$$

$$\lambda_c = \frac{2L}{n}$$

This eqn shows that disturbance dependence on y can not have any arbitrary value but an infinite no. of discrete value (wavelength).

For a given n-value λ_c is independent of λ_0 or corresponding ω.

Now $v_g = \frac{c}{\sin\theta}$ (phase vel. along z)

$$\lambda_g = \frac{v_g}{\nu} = \frac{v_g}{c/\lambda_0} = \frac{c/\sin\theta}{c/\lambda_0} = \frac{\lambda_0}{\sin\theta}$$

$$\cos\theta = \frac{\lambda_0}{\lambda_c}$$

$$\sin^2\theta + \cos^2\theta = \frac{\lambda_0^2}{\lambda_g^2} + \frac{\lambda_0^2}{\lambda_c^2} = 1$$

$$\frac{1}{\lambda_0^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2}$$

if $\lambda_c \to$ fixed $\quad \lambda_c = \frac{2\pi c}{\omega \cos\theta}$

and $\lambda_0 \to$ increased, λ_g will be increased, finally when $\lambda_0 \left(= \frac{2\pi c}{\omega}\right)$ becomes equal to λ_c, λ_g, v_g becomes infinity.

when $\lambda_0 > \lambda_c$, λ_g, v_g becomes imaginary $\qquad v_g$ imag $\to$ no wave flows

$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2} = -ve = -\frac{1}{k'^2}$$

$$\lambda_g = \pm i k \quad , \quad v_g = \frac{c \lambda_g}{\lambda_0}$$

$$e^{i\omega \sin\theta \, z/c} = e^{i\omega z/v_g} \quad = i k_1$$

$$= e^{-\frac{\omega z}{k_1}}$$

So, instead of propagate along z-axis, represent a decreasing exponential and hence attenuated wave.

This is cut off freq. / wavelength

For a particular mode, $\to$ it is the longest wavelength of a wave in free space, which can be propagated.

microwave $\to$ wavelength $\to$ laboratory dimension

$$\lambda_0 = \frac{2\pi c}{\omega}$$

$\lambda > \lambda_0 \to$ will be attenuated

Group Velocity. We have seen that the phase velocity $v_p = c/\sin\theta$ exceeds velocity of light. This apparently contracts the theory of relativity where we have postulated that no signal can be propagated with a velocity greater than the velocity of light. The resolution of the apparent difficulty is that the energy is propagated along the guide with a smaller velocity than the velocity of light, namely with the so called group velocity. *Signals are transmitted with the group velocity and not with the phase velocity. The phase velocity is not the velocity of propagation of any physical quantity : this simply represents the velocity of a point of constant phase on the wave.*

To determine the group velocity *i.e.* the velocity of energy propagation we shall calculate the time average energy density and time averaged-Poynting vector. Then dividing the Poynting vector (power flow) by the energy density, the velocity of energy propagation (group velocity) may be obtained.

In terms of λ_g and λ_c, the electric field (4) may be written as

$$\mathbf{E} = \hat{i} E' \sin\left(\frac{2\pi\lambda}{\lambda_c}\right) e^{i((2\pi z/\lambda_g) - \omega t)} \qquad \ldots(12)$$

(where $E' = 2iE_{01}$)

The magnetic field induction in the guide is readily obtained from

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{and assuming that } \mathbf{B}(\mathbf{r}, t) = B(\mathbf{r})\, e^{-i\omega t}$$

we quickly find

$$B(\mathbf{r}, t) = \hat{j} E' \frac{2\pi}{\omega\lambda_g} \sin\frac{2\pi y}{\lambda_c} e^{i}\,\{(2\pi z/\lambda_g) - \omega t\} + \hat{k} i E' \frac{2\pi}{\omega\lambda_g} \cos\frac{2\pi y}{\lambda_c} e^{i}\,\{(2\pi z/\lambda_g) - \omega t\} \qquad \ldots(13)$$

The time average Poynting vector is

$$\langle S_z \rangle = \tfrac{1}{2} R_e(\mathbf{E}^* \times \mathbf{H}) = \tfrac{1}{2} R_e(E_x^* H_y)$$

$$= \tfrac{1}{2} R_e\left[E'^* \sin\left(\frac{2\pi y}{\lambda_c}\right) \frac{1}{\mu_0} E' \left(\frac{2\pi}{\omega\lambda_g}\right) \sin\left(\frac{2\pi y}{\lambda_c}\right)\right]$$

$$= 2\, E'^* E' \left(\frac{2\pi}{\mu_0\omega\lambda_g}\right) \sin^2\left(\frac{2\pi y}{\lambda_c}\right)$$

Integrating this expression from $y = 0$ to $y = L$ across the guide yields the total average power travelling along the guide

$$\langle S_z \rangle = \int_0^L \langle S_z \rangle\, dy = \tfrac{1}{4} E'^* E' \left(\frac{2\pi}{\mu_0\omega\lambda_g}\right) L \qquad \ldots(14)$$

$$\left(\text{since } \int_0^L \sin^2 \frac{2\pi y}{\lambda_c}\, dy = \frac{L}{2}\right)$$

The . e averaged energy density is

$$\langle u \rangle = \tfrac{1}{4} \operatorname{Re}[\mathbf{E}^* \cdot \mathbf{D} + \mathbf{B}^* \cdot \mathbf{H}]$$

$$= \tfrac{1}{4} \operatorname{Re}\left[\epsilon_0 E'^* E' \sin^2\left(\frac{2\pi y}{\lambda_c}\right) + \frac{1}{\mu_0} E'^* E' \left(\frac{2\pi}{\omega\lambda_g}\right)^2 \sin^2\left(\frac{2\pi y}{\lambda_c}\right) + \frac{1}{\mu_0} E'^* E' \left(\frac{2\pi}{\omega\lambda_g}\right)^2 \cos^2\left(\frac{2\pi y}{\lambda_c}\right)\right] \qquad \ldots(15)$$

Integrating in y-direction from $y = 0$ to $y = L$ across the guide, effectively replaces $\sin^2\left(\frac{2\pi y}{\lambda_c}\right)$ and $\cos^2\left(\frac{2\pi y}{\lambda_c}\right)$ by $\frac{L}{2}$. Thus

$$<S> = \int_0^L <u>\, dy = \frac{1}{4} E'^* E' \frac{L}{2}\left[\varepsilon_0 + \frac{4\pi}{\mu_0\omega^2}\left(\frac{1}{\lambda_g^2}+\frac{1}{\lambda_c^2}\right)\right]$$

$$= \frac{1}{4} E'^* E' \frac{L}{2}\left[\varepsilon_0 + \frac{4\pi^2}{\mu_0\omega^2\lambda_0^2}\right]$$

$$= \frac{1}{4} E'^* E' \frac{L}{2}\left[\varepsilon_0 + \frac{1}{\mu_0 c^2}\right] = \frac{1}{4} E'^* E' \frac{L}{2}[\varepsilon_0 + \varepsilon_0]$$

$$= \frac{1}{4} E'^* E' \varepsilon_0 L \qquad ...(16)$$

The velocity of *energy propagation* or *group velocity*

$$v_G = \frac{<S>}{<U>} = \frac{\frac{1}{4} E'^* E'\left(\frac{2\pi}{\mu_0\,\omega\lambda_g}\right)L}{\frac{1}{4} E'^* E\varepsilon_0 L} = \frac{2\pi}{\mu_0\varepsilon_0\omega\lambda_g}$$

$$= \frac{2\pi c^2}{\omega\lambda_g} = c\frac{\lambda_0}{\lambda_g} = c\sin\theta \qquad ...(17)$$

Obviously group velocity v_G is smaller than the velocity of light.

The group velocity v_G may be obtained by usual relation $v_G = \frac{\partial\omega}{\partial \mathbf{k}}$, the ratio $\frac{\omega}{\mathbf{k}}$ is called the phase velocity which in this case becomes

$$v_c = \frac{\partial\omega}{\partial k_g}. \qquad ...(18)$$

Equation (11) may be expressed as

$$k^2 = k_c^2 + k_g^2 \qquad (\text{as } k = 2\pi/\lambda).$$

In free space $k = \omega/c$, therefore

$$\omega^2/c^2 = k_c^2 + k_g^2$$

or

$$\{\omega = c\sqrt{(k_c^2 + k_g^2)}\}. \qquad ....(19)$$

Differentiating this equation with respect to k_g, we get

$$v_G = \frac{\partial\omega}{\partial k_g} = c\,.\,\frac{1}{2}(k_c^2+k_g^2)^{-1/2}\times 2K_g = \frac{ck_g}{\omega/c} \text{ using(19)}$$

$$= c^2.\left(\frac{k_g}{\omega}\right) = \frac{c^2}{c/\sin\theta} = c\sin\theta. \qquad ...(20)$$

We have $v_g = c/\sin\theta$ and $v_G = c\sin\theta$.

Thus it is readily apparent.

$$v_g v_G = c^2. \qquad ...(21)$$

9.12. Wave Guides

A wave guide is a hollow pipe of infinite extent. We shall now consider the propagation of electromagnetic waves along a hollow conducting pipe (wave guide) of arbitrary (simply connected) cross section uniform along its length. We shall assume that the walls of the wave guide are perfectly conducting and take the z-axis along the wave guide. For simplicity we further assume that the interior of wave guide is vacuum. As shown in chapter 7, each component of **E** and **B** must satisfy wave equation which in vacuum has the form

$$\left.\begin{aligned}\nabla^2\mathbf{E}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} &= 0 \text{ or } \nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad ...(a)\\ \nabla^2\mathbf{B}-\mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2} &= 0 \text{ or } \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \quad ...(b)\end{aligned}\right\} \quad ...(1)$$

In addition to these wave equations Maxwell's equations and boundary conditions must be satisfied. As the wave is propagated along Z-axis, we assume solutions of the form

$$\left.\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}(x,y)\,e^{ik_z z-i\omega t} \quad ...(a)\\ \mathbf{B}(\mathbf{r},t) &= \mathbf{B}(x,y)\,e^{ik_z z-i\omega t} \quad ...(b)\end{aligned}\right\} \quad ...(2)$$

Appropriate linear combinations can be formed to give travelling or standing waves in the Z-direction. The wave number k_z is known from preceding section but however we assume that it is unknown parameter which may be real or complex. With this assumed Z dependence of fields, the wave equation reduces to two dimensional form

$$\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}-k_z^2\mathbf{E}+\frac{\omega^2}{c^2}\mathbf{E} = 0$$

$$\left.\begin{aligned}\text{or}\quad &\frac{\partial^2\mathbf{E}}{\partial x^2}+\frac{\partial^2\mathbf{E}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_z^2\right]\mathbf{E} = 0 \quad ...(a)\\ \text{and}\quad &\frac{\partial^2\mathbf{B}}{\partial x^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_z^2\right]\mathbf{B} = 0 \quad ...(b)\end{aligned}\right\} \quad ...(3)$$

Now if $\nabla_\perp^2$ is transverse part of Laplacing operator ∇^2, then

$$\nabla_\perp^2 = \nabla^2-\frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad ...(4)$$

In these notations equation (3) takes the form

$$\left[\nabla_\perp^2+\left(\frac{\omega^2}{c^2}-k_z^2\right)\right]\begin{Bmatrix}\mathbf{E}\\ \mathbf{B}\end{Bmatrix} = 0 \quad ...(5)$$

Now Maxwell's curl equations for free space are

$$\left.\begin{aligned}&\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad (a)\\ \text{and}\quad &\nabla\times\mathbf{B} = \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t} \text{ or } \nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad ...(b)\end{aligned}\right\} \quad ...(6)$$

which in terms of components can be written as

$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t}\\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t}\end{aligned}\right] \quad ...(7a)$$

and
$$\left.\begin{aligned}\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z} &= \frac{1}{c^2}\frac{\partial E_x}{\partial t}\\ \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x} &= \frac{1}{c^2}\frac{\partial E_y}{\partial t}\\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y} &= \frac{1}{c^2}\frac{\partial E_z}{\partial t}\end{aligned}\right] \qquad ...(7b)$$

From the form of exponential $e^{ik_g z - i\omega t}$ it is apparent that
$$\frac{\partial}{\partial z} \rightarrow ik_g \text{ and } \frac{\partial}{\partial t} \rightarrow -i\omega$$

Therefore equations (7) lead to
$$\left.\begin{aligned}\frac{\partial E_z}{\partial y} - ik_g E_y &= i\omega B_x & ....(a)\\ ik_g E_x - \frac{\partial E_z}{\partial y} &= i\omega B_y & ....(b)\\ \frac{\partial \mathbf{E}_y}{\partial x} - \frac{\partial \mathbf{E}_x}{\partial y} &= i\omega B_z & ...(c)\end{aligned}\right] \qquad ...(8)$$

and
$$\left.\begin{aligned}\frac{\partial B_z}{\partial y} - ik_g B_y &= -\frac{i\omega}{c^2}\mathbf{E}_x & ....(a)\\ ik_g B_x - \frac{\partial B_z}{\partial x} &= -\frac{i\omega}{c^2}\mathbf{E}_y & ....(b)\\ \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= -\frac{i\omega}{c^2}\mathbf{E}_z & ...(c)\end{aligned}\right] \qquad ...(9)$$

Substituting values of B_y from (8b) in (9a), we get
$$\frac{\partial B_z}{\partial y} - ik_g\left(\frac{ik_g\mathbf{E}_x - \dfrac{\partial \mathbf{E}_z}{\partial x}}{i\omega}\right) = -\frac{i\omega}{c^2}\mathbf{E}_x$$

i.e.
$$\frac{\partial B_z}{\partial y} + \frac{k_g}{\omega}\frac{\partial \mathbf{E}_z}{\partial x} = \left(\frac{ik_g^2}{\omega} - \frac{i\omega}{c^2}\right)\mathbf{E}_x$$

i.e.
$$\mathbf{E}_x = \frac{i}{\left(\dfrac{\omega^2}{c^2} - k_g^2\right)}\left[k_g\frac{\partial \mathbf{E}_z}{\partial x} - \omega\frac{\partial B_z}{\partial y}\right] \qquad ...(10)$$

Now eliminating B_x from (8a) and (9b), we get
$$\mathbf{E}_y = \frac{i}{\left[\dfrac{\omega^2}{c^2} - k_g^2\right]}\left[k_g\frac{\partial \mathbf{E}_z}{\partial q} - \omega\frac{\partial B_z}{\partial x}\right] \qquad ...(11)$$

Similarly eliminating $\mathbf{E}_y$ and $\mathbf{E}_x$ in turn from (8) and (9), we get

$$B_x = \frac{i}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\left(k_g \frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial \mathbf{E}_z}{\partial y}\right) \quad \text{...(12)}$$

$$B_y = \frac{i}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\left(k_g \frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial \mathbf{E}_z}{\partial x}\right) \quad \text{...(13)}$$

Equation (10), (11), (12) and (13) show that it is sufficient to determine $\mathbf{E}_z$ and B_z as the appropriate solutions of the two dimensional wave equation (5). The other components can then be calculated from (10), (11), (12) and (13). In general cylindrical guide with perfectly conducting walls, such as that shown in fig. 9.11, is under consideration, then the appropriate boundary conditions are that the tangential component of **E** and the normal component of **B** should vanish on the surface of the conductor. The boundary condition for two dimensional wave equation for $\mathbf{E}_z$ viz.

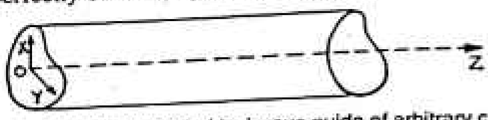


Fig. 9.11 Hollow cylindrical wave guide of arbitrary cross sectional shape

$$\nabla_\perp^2 \mathbf{E}_z + \left(\frac{\omega^2}{c^2} - k_g^2\right)\mathbf{E}_z = 0 \quad \text{...(14)}$$

that the tangential component of **E** should vanish on the surface of the conductor requires

$$\mathbf{B}_z \big|_S = 0 \quad \text{...(15)}$$

The normal component of **B** at the surface requires

$$|\mathbf{B} \cdot \mathbf{n}|_S = 0$$

where **n** is unit vector normal the surface

i.e.

$$\left| B, (x, y)\, e^{ik_g z - i\omega t} \cdot \mathbf{n} \right|_S = 0$$

this condition implies

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \quad \text{...(16)}$$

where $\partial/\partial x$ is the normal derivative at the point on the surface. The two dimensional wave equations (5) for $\mathbf{E}_z$ and B_z together with the boundary conditions on $\mathbf{E}_z$ and B_z at the surface of the cylinder allow only certain values of axial wave number k_g for a given frequency ω. Because of different boundary conditions on $\mathbf{E}_z$ and B_z they can not be generally satisfied simultaneously. Consequently the fields may be divided into two distinct classes.

Transverse Magnetic (TM) Mode. In this case $B_z = 0$ always *i.e.* magnetic fields is always perpendicular to the direction of propagation; hence the name transverse magnetic. These type of wave are also sometimes named as *electric type waves* or **E** waves.

Substituting $B_z = 0$, equations (10) to (13) take to form

$$\mathbf{E}_x = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)}\frac{\partial \mathbf{E}_z}{\partial x} = \frac{ik_g}{(k_o^2 - k_g^2)}\frac{\partial \mathbf{E}_z}{\partial x} \quad \text{...(17)}$$

$$\mathbf{E_y} = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{E_z}}{\partial y} = \frac{ik_g}{(k_o^2 - k_g^2)} \frac{\partial \mathbf{E_z}}{\partial y} \qquad ...(18)$$

$$\mathbf{B_x} = -\frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right)c^2} \frac{\partial \mathbf{E_z}}{\partial y} = -\frac{i\omega}{(k_o^2 - k_g^2)\,c^2} \frac{\partial \mathbf{E_z}}{\partial y} \qquad ...(19)$$

$$\mathbf{B_y} = \frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right)c^2} \frac{\partial \mathbf{E_z}}{\partial y} = \frac{i\omega}{(k_o^2 - k_g^2)\,c^2} \frac{\partial \mathbf{E_z}}{\partial x} \qquad ...(20)$$

where $k_0 = \frac{\omega}{c}$ = free space wave number

Thus in *TM* mode, all the transverse components of **E** and **B** can be expressed in terms of longitudinal component of the electric field. This component may be obtained by solving the two dimensional wave equation (5) for $\mathbf{E_z}$ *viz.*

$$\nabla_\perp^2 \mathbf{E_z} + \left(\frac{\omega^2}{c^2} - k_g^2\right)\mathbf{E_z} = 0 \qquad ...(21)$$

or

$$\nabla_\perp^2 \mathbf{E_z} + (k_0^2 - k_g^2)\,\mathbf{E_z} = 0$$

The boundary condition for this equation is $\qquad \mathbf{E_z}\,|_S = 0 \qquad ...(i)$

(ii) Transverse electric (TE) Mode. In this case $\mathbf{E_z} = 0$ always *i.e.* electric field is always perpendicular to the direction of propagation of the wave, hence the name *transverse electric mode*. These types of waves are also called magnetic type waves or *H*-waves.

Substituting $\mathbf{E_z} = 0$, equations (10) to (13) take the form

$$\mathbf{E_x} = \frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{B_z}}{\partial y} = \frac{i\omega}{k_0^2 - k_g^2} \frac{\partial \mathbf{B_z}}{\partial y} \qquad ...(22)$$

$$\mathbf{E_y} = -\frac{i\omega}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial \mathbf{B_z}}{\partial x} = -\frac{i\omega}{k_0^2 - k_g^2} \frac{\partial \mathbf{B_z}}{\partial y} \qquad ...(23)$$

$$B_z = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial B_z}{\partial x} = -\frac{ik_g}{k_0^2 - k_g^2} \frac{\partial B_z}{\partial x} \qquad ...(24)$$

$$B_y = \frac{ik_g}{\left(\frac{\omega^2}{c^2} - k_g^2\right)} \frac{\partial B_z}{\partial y} = -\frac{ik_g}{k_0^2 - k_g^2} \frac{\partial B_z}{\partial y} \qquad ...(25)$$

Thus in *TE* mode all transverse components of **E** and **B** can be expressed in terms of longitudinal component of magnetic field. This component may be obtained by solving two dimensional wave equation (5) for B_z *viz.*

$$\nabla_{\perp}^{2} B_z + \left(\frac{\omega^2}{c^2} - k_g^2\right) B_z = 0$$

or

$$\nabla_{\perp}^{2} B_z + (k_0^2 - k_g^2) B_z = 0 \qquad ...(26)$$

with the boundary condition

$$\left.\frac{\partial B_z}{\partial n}\right|_S = 0 \qquad ...(i)$$

According to equations (22)–(25), this condition ensures that normal component of **B** is zero.

Thus the problem of determining the electromagnetic field in a wave-guide reduces to that of finding solutions of two dimensional wave equation

$$[\nabla_{\perp}^{2} + (k_o^2 - k_g^2)]\, \psi = 0 \text{ or } [\nabla_{\perp}^{2} + k_c^2]\, \psi = 0 \qquad ...(27)$$

subject to boundary conditions $\psi|_S = 0$ and $\partial\psi/\partial n\, |_S = 0$.

Here
$$k_c^2 = k_0^2 - k_g^2 \text{ or } k_0^2 = k_c^2 + k_g^2 \qquad ...(28a)$$

Equation (28 a) is equivalent to
$$\frac{1}{\lambda_g^2} = \frac{1}{\lambda_c^2} + \frac{1}{\lambda_g^2} \qquad ...(28\text{ b})$$

which is same as equation (11) of preceding section 9.11 Thus the conclusions drawn in the preceding section are quite general and are not limited to the case of propagation between parallel conducting planes.

For a given cross section the solution ψ exists only for a certain definite eigen values of the parameter k_c^2. It is easy to see that the constant k_c^2 must be non-negative, since roughly speaking ψ must be oscillatory in order to satisfy required boundary condition on opposite sides of the cylinder. These will from a spectrum of eigen values $(k_c^2)_\lambda$ and corresponding solutions ψ_λ ($\lambda = 1, 2, 3$) form an orthogonal set. These different solutions are called the *modes of the guide*. For a given frequency ω the wave number k_g is determined for each value of λ.

$$(k_g^2)_\lambda = k_0^2 - (k_c^2)_\lambda = \omega^2/c^2 - (k_c^2)_\lambda \qquad ...(29)$$

Since $k_g^2 = k_0^2 - k_c^2$, therefore we note that for $k_0 > k_c$ or $\omega > \omega_c$, the wave number k_g is real ; hence waves of such modes can propagate in the guide. But if $k_c < k_c$ or $\omega < \omega_c$, k_g is imaginary which is turn implies the attenuation of **E** and **B** given by (2) : hence such modes can not propagate and are called *cut off modes*. The frequency given by

$$(\omega_c)\lambda = [c]\,(k_c) \qquad ...(30)$$

is called cut off frequency. Then the *wave number* can be written

as
$$(k_g)\lambda = \frac{1}{c}\sqrt{[\omega^2 - (\omega_c^2)\,\lambda]} \qquad ...(31)$$

From this it follows that a guide acts a sort of high pass filter in the sense that only frequencies greater that cut off frequency can be propagated in the guide. Moreover at any given frequency only a finite mumber of modes can propagate. It is often convenient to choose the dimensions of the guide so that at the operating frequency only the lowest mode can occur.

The velocity of propagation *of the wave (i.e. group velocity) along the wave guide* is given by the derivative

$$v_z = \frac{\partial\omega}{\partial k_g} = \frac{\partial}{\partial k_g} c\,(k_g^2 + k_c^2)^{1/2}$$

$$= \frac{ck_g}{\sqrt{(k_0^2 + k_c^2)}} = \frac{ck_g}{k_0} = \frac{c^2 k_g}{\omega} \quad ...(32)$$

For a given k_c : this varies from o to c when k_g varies from o to ∞.

The **phase velocity** v_g in the guide is given by

$$v_g = \frac{\omega}{k_g} = \frac{\omega}{\sqrt{(\omega^2 - \omega_c^2)/c}} = \frac{c}{\sqrt{\left[1 - \left(\frac{\omega_c^2}{\omega^2}\right)\right]}}$$

Obviously $v_g > c$ and becomes infinite exactly at cut off frequency. The energy in the guide is propagated with the group velocity.

***TEM* Waves.** In addition to *TE* and *TM* modes, there is a degenerate mode, called the transverse electromagnetic (*TEM*) mode, in which both E_z and B_z vanish. If we substitute $E_z = B_z = 0$ in equations (10)–(13), we note that there is no non-zero component of **E** or **B**. This implies that ***TEM* wave can not be propagated along the wave-guide.**

9.13. Rectangular Wave Guide

The most commonly used wave guide is that of rectangular cross-section having inner dimension a and b as shown in fig. 9.21.

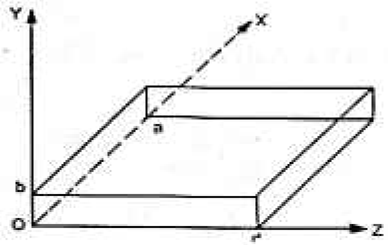


Fig. 9.12

The solution of two dimensional wave equation

$$(\nabla_\perp^2 \neq k_c^2)\,\psi = 0 \quad ...(1)$$

can be carried out in rectangular coordinates as follows :

***TE* Mode.** For *TE* mode $E_z = 0$: hence equation (1) is to be written for B_z ; which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) B_z = 0 \quad ...(2)$$

The boundary conditions are

$$\left.\frac{\partial H_z}{\partial n}\right|_S = 0$$

i.e.

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x = 0 \text{ and } x = a$$

and

$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y = 0 \text{ and } y = b$$

We shall solve equation (2) by the method of separation of variables.

Therefore writing $\quad B_z(x, y) = X(x)\,Y(y) = XY \quad ...(3)$

where X is a function of x only and Y is a function of y only.

Susbtituting equation (3) in (2) and dividing by XY, we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

or $$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+k_c^2 = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

In above equation L.H.S. is a function of x only, while R.H.S. is a function of y only. Hence this equation will be satisfied if both sides are equal to a constant say p^2 *i.e.*

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}+k_c^2 = +p^2$$

or $$\frac{\partial^2 X}{\partial x^2}+(k_c^2-p^2)X = 0$$

or $$\frac{\partial^2 X}{\partial x^2}+q^2 X = 0 \qquad ...(4)$$

where $$q^2 = k_c^2-p^2 \qquad ...(5)$$

and $$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = p^2$$

or $$\frac{\partial^2 Y}{\partial y^2}+p^2 Y = 0 \qquad ...(6)$$

The solution of equation (4) and (6) are

$$X(x) = A\cos qx + B\sin qx \qquad ...(7)$$
$$Y(y) = C\cos py + D\sin py \qquad ...(8)$$

where A, B, C, D are arbitrary constants.

We have the boundary conditions

$$\frac{\partial B_z}{\partial x} = 0 \text{ at } x=0 \text{ and } x=a$$
$$\frac{\partial B_z}{\partial y} = 0 \text{ at } y=0 \text{ and } y=b$$

These conditions are equivalent to

$$\frac{\partial X}{\partial x} = 0 \text{ at } x=0 \text{ and } x=a$$

and $$\frac{\partial Y}{\partial y} = 0 \text{ at } y=0 \text{ and } y=b.$$

Differentiating equations (7) and (8), we get

$$\frac{\partial X}{\partial x} = -Aq\sin qx + Bq\cos qx$$
$$\frac{\partial Y}{\partial y} = -Cp\sin qy + Dp\cos py$$

Applying boundary condition $\left.\frac{\partial Z}{\partial x}\right|_{x=a} = 0$, we get

$Bq = 0 \therefore$ This gives $B = 0$

Now applying boundary condition $\left.\frac{\partial X}{\partial x}\right|_{x=a} = 0$, we get

$$-Aq\sin qa = 0$$

We must take $A \neq 0$ since otherwise $X = 0$ and $B_z = 0$. Hence

$$\sin qa = 0 \text{ or } qa = m\pi, \text{ that is,}$$

$$q = \frac{m\pi}{a} \quad (m \text{ integer}) \qquad ...(11)$$

In precisely the same manner we conclude that $D = 0$ and p must be restricted to values $p = n\pi/b$ where n is an integer. In this way be obtain the solutions

$$X(x) = A\cos\left(\frac{m\pi}{a}\right)x\,;\quad Y(y) = C\cos\left(\frac{n\pi}{b}\right)y \qquad ...(12a)$$

where

$$m = 1, 2, 3, ...\quad n = 1, 2, 3,....$$

and

$$k_c^2 = p^2 + q^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

The solution for $B_z(x, y)$ is consequently

$$B_z(x, y) = B_0 \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad ...(13a)$$

when

$$(k_c^2)_{mn} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^{1/2} \qquad ...(13b)$$

Here the indices mn specify the mode. The *cut off frequency* ω_{mn} is given by

$$\omega_{mn} = \pi c\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} \qquad ...(14)$$

The modes corresponding to m and n are represented as TE_{mn}. The case $m = n = 0$ gives a static field which does not represent a wave propagation ; hence the mode TE_{00} represents non-trivial solution. If $a > b$, the lowest *cut off frequency* results for $m = 1$ and $n = 0$,

i.e.

$$\omega_{10} = \frac{\pi c}{a} \quad \text{or} \quad (k_c)_{10} = \frac{\pi}{a} \qquad ...(15)$$

The mode (TE_{10}) represents the *dominant TE mode* and is the one used in most practical situations. The values E_x, E_y, B_x and B_y for TE mode may be obtained from equations (22)–(25) of preceding section by substituting the solution for B_z, which is

$$B_z(r, t) = B_z(x, y)\, e^{ik_g z - i\omega t}$$

$$= B_0\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\, e^{ik_g z - i\omega t} \qquad ...(16)$$

Thus we have

$$\left.\begin{aligned} E_x &= \frac{in\pi\omega}{k_c^2 b} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ E_y &= \frac{im\pi\omega}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_x &= -\frac{im\pi k_g}{k_c^2 a} B_0 \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{ik_g z - i\omega t} \\ B_y &= -\frac{in\pi k_g}{k_c^2 a} B_0 \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{ik_g z - i\omega t} \end{aligned}\right\} \quad ...(17)$$

For *TE* mode, these equations yield

$$\left(\text{put } m = 1,\ n = 0, k_c^2 = \frac{\pi^2}{a^2}\right)$$

$$\left.\begin{aligned} E_x = B_y = E_z = 0\,;\ B_z &= B_0 \cos\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ \text{and} \quad B_x &= -\frac{ik_g a}{a} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \\ E_y &= \frac{i\omega a}{\pi c} B_0 \sin\left(\frac{\pi x}{a}\right) e^{ik_g z - i\omega t} \end{aligned}\right\} \quad ...(18)$$

The presence of a factor i in B_x (and E_y) means that there is a spatial (or temporal) phase difference of $\pi/2$ between B_x (and E_y) and B_z in the propagation.

***TM* Mode :** For *TM* mode $B_z = 0$; hence equation (1) is to be written for E_z which takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) E_z = 0 \quad ...(19)$$

The boundary conditions are $E_z = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$.

Solving equation (19) as for *TE* case, we note that the solution of equation (19) is of the form

$$E_z(x, y) = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \quad ...(20)$$

where k_c^2 and hence ω_{mn} are still given by equations (12b) and (14). This implies that *TE and TM modes of a rectangular guide have the same set of cut off frequencies.* However in this case $m = 1$ and $n = 0$ represents non-trival solution since this gives $E_z = 0$ and hence all components of **E** and **B** will be zero.

It is obvious that in this case the lowest mode has $m = n = 1$ and may be represented by TM_{11}. The **cut off frequency of lowest mode** is given by

$$\omega_{11} = \pi c \left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = \frac{\pi c}{a}\left[1 + \frac{a^2}{b^2}\right]^{1/2} \quad ...(21)$$

Since $a < b$, therefore the *cut off frequency of lowest TM mode is greater than that of the lowest TE mode by the factor* $\left[1 + \left(\frac{a^2}{b^2}\right)\right]^{1/2}$ The fields E_x, E_y, B_x, B_y for *TM* mode may be obtained from equations (17)–(22) of preceding section if we substitute

$$E_z(\mathbf{r}, t) = E_0(x, y)\, e^{ik_g z - i\omega t}$$

or
$$E_z(\mathbf{r}, t) = E_0 \sin\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{-ik_g z - i\omega t} \quad ...(22)$$

Thus we have

$$\left.\begin{aligned} E_x &= \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ E_y &= \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_x &= -\frac{i\omega n\pi}{k_c^2 bc} E_0 \sin\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \\ B_y &= \frac{i\omega n\pi}{k_c^2 ac^2} E_0 \cos\left[\frac{m\pi x}{a}\right] \sin\left[\frac{n\pi y}{b}\right] e^{ik_g z - i\omega t} \end{aligned}\right\} \quad ...(22)$$

Ex. 2. *What must be the width of a rectangular guide such that the energy of electromagnetic radiation, whose free space wavelength is 3·0 cm, travels down the guide at 95% of the speed of light ?* **(Meerut 1970)**

Solution. We know that the energy of electromagnetic radiation is propagated down the guide with *group velocity* given by

$$v_z = \frac{ek_g}{k_0} = \frac{c\sqrt{(k_0^2 - k_c^2)}}{k_0} = c\left[1 - \frac{k_c^2}{k_0^2}\right]^{1/2} \quad ...(1)$$

For rectangular wave guide

$$(k_c^2)_{mn} = \pi^2\left[\frac{m^2}{a^2} + \frac{n^2}{a^2}\right]$$

For *TE mode* and $a > b$; the maximum cut off frequency is given by $m = 1$ and $m = 0$,

$$(k_c^2)_{10} = \pi^2\left[\frac{1}{a^2} + \frac{0}{a^2}\right] = \frac{\pi^2}{a^2} \quad ...(2)$$

If λ_0 is *free space wavelength,* then
$$k_c = 2\pi/\lambda_0 \quad ...(3)$$

Using (2) and (3), equation (1) gives

$$v_z = c\left[1 - \frac{(\pi^2/a^2)}{(2\pi/\lambda c)^2}\right]^{1/2} = c\left[1 - \frac{\lambda_0^2}{4\pi^2}\right]^{1/2}$$

Squaring, we get

$$v_z^2 = c^2\left[1 - \frac{\lambda_0^2}{4a^2}\right]$$

or
$$\frac{v_z^2}{c^2} = 1 - \frac{\lambda_0^2}{4a^2}$$

This gives

$$a = \frac{\lambda_0}{2\left[1 - \frac{v_z^2}{c^2}\right]^{1/2}}$$

Here $v_z = \frac{95}{100}c = 0.95\,c$ and $\lambda_0 = 3$ cm $= 0.03$ m

$$\therefore \qquad a = \frac{0.03}{2\left[1-\left(\frac{0.95c}{c}\right)^2\right]^{1/2}} = \frac{0.03}{2\,[1-(0.95)^2]^{1/2}}$$

$$= \frac{0.03}{2\times 0.12} = 0.048 \text{ m} = 4.8 \text{ cm.}$$

9.14. Circular Wave Guide

The next most important example of wave guides after the rectangular guide is that in which the bounding surfaces are circular cylinders. This includes both the circular guide and the coaxial line, which has two concentric cylinder, with the wave propagated in the angular space between them. In either case the two dimensional wave equation *viz.*

$$(\nabla_\perp^2 + k_c^2)\,\psi = 0 \qquad \text{...(1)}$$

is to be solved in cylinderical coordinates, but here we shall restrict ourselves for cases where cross-section is circular. Let a be the radius of circular cross-section

We have

$$\nabla_\perp^2 = \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]$$

Therefore two dimensional wave equation (1) in this case becomes

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right]\psi + k_c^2\,\psi = 0 \qquad \text{...(2)}$$

If a is the radius of circular cross-section, then the boundary conditions are

$$\psi\Big|_{r=a} = 0 \text{ and } \frac{\partial\psi}{\partial n}\Big|_{r=a} = 0 \qquad \text{for all values of } \theta \qquad \text{...(3)}$$

Now let us write Maxwell's equations in cylinderical coordinates (r, θ, z). Maxwell's curl equations in free space are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

and

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \text{...(4)}$$

In cylindrical coordinates equations take the form

$$\left(\frac{1}{r}\frac{\partial E_z}{\partial\theta} - \frac{\partial E_\theta}{\partial z}\right)\hat{n}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r}\right)\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,E_\theta) - \frac{\partial E_r}{\partial\theta}\right]\hat{n}$$

$$= -\left[\frac{\partial B_r}{\partial t}\hat{n}_r + \frac{\partial B_\theta}{\partial t}\hat{n}_\theta + \frac{\partial B_z}{\partial t}\hat{n}_z\right]$$

and

$$\left[\frac{1}{r}\frac{\partial B_z}{\partial\theta} - \frac{\partial B_\theta}{\partial r}\right]\hat{n}_r + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\hat{n}_r + \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,B_\theta) - \frac{\partial B_r}{\partial\theta}\right]\hat{n}_z$$

$$= \frac{1}{c^2}\left[\frac{\partial E_r}{\partial t}\hat{n}_r + \frac{\partial E_\theta}{\partial t}\hat{n}_\theta + \frac{\partial E_z}{\partial t}\hat{n}_z\right]$$

Comparing coefficients of $\hat{n}_r$, $\hat{n}_\theta$ and $\hat{n}_z$ on either sides, we get